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LETTER TO THE EDITOR

Aperiodic interactions on hierarchical lattices: an exact criterion for the Potts ferromagnet criticality

A C N de Magalhães†§, S R Salinas‡‖ and C Tsallis†¶

† Centro Brasileiro de Pesquisas Físicas—CBPF/CNPq Rua Dr Xavier Sigaud, 150, 22290-180, Rio de Janeiro, RJ, Brazil

‡ Instituto de Física, Universidade de São Paulo, Caixa Postal 66318, 05315-970, São Paulo, SP, Brazil

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Abstract. We discuss the critical behaviour of the q-state Potts model on a diamond-like hierarchical lattice with ferromagnetic interactions according to an aperiodic two-letter substitutional sequence. We show that the geometric (deterministic) fluctuations become relevant for $\omega > 1 - D/(2 - \alpha_u)$, where ω is the wandering exponent of the sequence, D is the fractal dimension of the lattice, and α_u is the critical exponent associated with the specific heat of the uniform model. We also point out that the criteria for analysing the relevance of deterministic and random fluctuations are generically *different*.

The introduction of quenched disorder is known to change the critical behaviour of ferromagnetic systems whenever (but not only) the corresponding uniform model is characterized by a positive exponent α_u associated with the divergence of the specific heat [1, 2]. A similar effect may be anticipated if the exchange interactions are chosen according to an aperiodic, although deterministic, type of rule. Recently, Luck [3] proposed a heuristic criterion which indicates indeed that the geometric fluctuations produced by the aperiodic rule may be responsible for changing the nature of the critical behaviour.

The discovery of quasicrystals [4] motivated the investigation of different types of spin models with aperiodic interactions. Recent calculations for the ground state of a quantum Ising chain support the heuristic criterion of Luck [3, 5]. In previous papers [6], one of us has taken advantage of the simplicity of diamond-type hierarchical lattices (DHL) [7, 8] to analyse the critical behaviour of the Ising model with a distribution of ferromagnetic exchange interactions according to a certain class of two-letter substitutional sequences. In this letter, we extend these results to the q-state Potts model with aperiodic ferromagnetic interactions on a general DHL, and derive an exact criterion to show the relevance of the geometric fluctuations above a critical number of states q_d. We also establish some contacts with calculations for the disordered Potts model.

The q-state Potts ferromagnet is given by the Hamiltonian

$$\mathcal{H} = -q \sum_{(i,j)} J_{ij} \delta_{\sigma_i,\sigma_j} \tag{1}$$

§ E-mail address: aglae@cat.cbpf.br

‖ E-mail address: ssalinas@if.usp.br

¶ E-mail address: tsallis@cat.cbpf.br

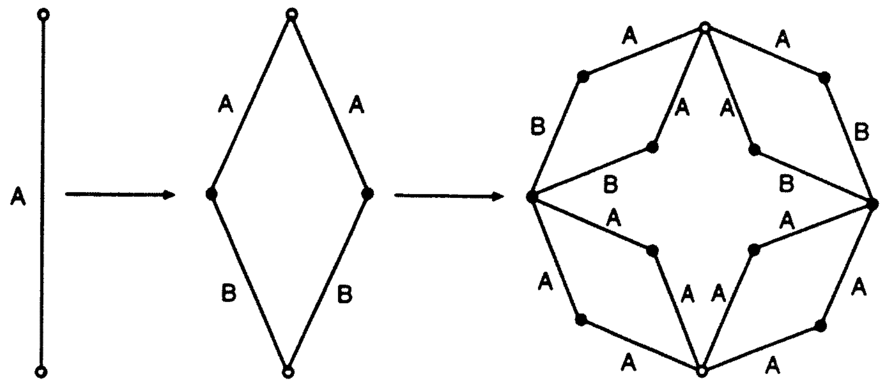

Figure 1. Some stages of the construction of a DHL with chemical length $b = 2$ and $m = 2$ branches (the simple diamond lattice) for the period-doubling sequence $(AB) \to (AB, AA)$ (letters A and B indicate the exchange interactions, $J_A > 0$ and $J_B > 0$).

where $\sigma_i = 1, 2, \ldots, q$ for all sites of a lattice, $J_{ij} > 0$, and the sum (i, j) refers to nearest-neighbour sites. To give an example, let us consider the simple diamond lattice (that is, a DHL with $m = 2$ branches in parallel, each of them with $b = 2$ bonds in series), and choose the ferromagnetic interactions J_{ij} according to the two-letter generalized Fibonacci sequence given by the substitutions $(A, B) \to (AB, AA)$, as indicated in figure 1 (to mimic a layered structure, the interactions are *aperiodic along the branches* of the lattice).

At each generation associated with the period-doubling sequence $(A, B) \to (AB, AA)$ the numbers N'_A and N'_B, of letters A and B, can be obtained from those of the preceding level, N_A and N_B, from the recursion relations

$$\begin{pmatrix} N'_A \\ N'_B \end{pmatrix} = \mathbf{M} \begin{pmatrix} N_A \\ N_B \end{pmatrix} \tag{2}$$

with the substitution matrix

$$\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \tag{3}$$

whose eigenvalues are $\lambda_1 = b = 2$ and $\lambda_2 = -1$. The total number of letters, $N^{(n)}$, at a large order n of the sequence construction, fluctuates asymptotically as $\Delta N^{(n)} \sim (N^{(n)})^{\omega}$, where

$$\omega = \frac{\ln |\lambda_2|}{\ln \lambda_1} \tag{4}$$

is the *wandering exponent* [3] of the geometric fluctuations.

Introducing the transmissivity variable [9],

$$t = \frac{1 - \exp(-q\beta J)}{1 + (q-1)\exp(-q\beta J)} \tag{5}$$

and using the break-collapse techniques [10], it is straightforward to write the exact recursion relations

$$t'_A = \frac{2t_A t_B + (q-2)t_A^2 t_B^2}{1 + (q-1)t_A^2 t_B^2} \tag{6}$$

and

$$t'_B = \frac{2t_A^2 + (q-2)t_A^4}{1 + (q-1)t_A^4} \tag{7}$$

where t_A and t_B are associated with $J_A > 0$ and $J_B > 0$, respectively. Now it is easy to show that the only physical fixed points are along the diagonal $t_A = t_B$ of the parameter space. There are two trivial stable fixed points ($t_A = t_B = 0$, and $t_A = t_B = 1$), and the non-trivial uniform fixed point, $0 < t_A = t_B = t_u^*(q) < 1$, where $t_u^*(q)$ comes from the equation $(q-1)(t_u^*)^3 + (t_u^*)^2 + t_u^* - 1 = 0$ (the function $t_u^*(q)$ decreases monotonically from 1 to 0 as q varies from 0 to ∞). The linearization in the neighbourhood of this uniform fixed point yields the matrix relation

$$\begin{pmatrix} \Delta t'_A \\ \Delta t'_B \end{pmatrix} = C(q)\mathbf{M}^T \begin{pmatrix} \Delta t_A \\ \Delta t_B \end{pmatrix} \tag{8}$$

where the prefactor $C(q)$ depends on q but does not depend on the particular two-letter sequence, and $\mathbf{M}^T$ is the transpose of the substitution matrix. The eigenvalues of this transformation are $\Lambda_1(q) = C(q)\lambda_1 = 2C(q)$ and $\Lambda_2(q) = C(q)\lambda_2 = -C(q)$. The expression for the largest eigenvalue, $\Lambda_1(q)$, also corresponds to the thermal eigenvalue of the linearization about the non-trivial fixed point of the corresponding uniform model (that is, with $J_A = J_B > 0$). Therefore, it is straightforward to write an expression for $C(q)$. For $\Lambda_1 = 2C(q) > 1$ (as in the uniform model), and $|\Lambda_2(q)| = C(q) < 1$, the fixed point in the (t_A, t_B) parameter space is of a hyperbolic character as illustrated in figure 2(a) (which indicates the existence of a critical line in the phase diagram in terms of the temperature and the ratio $r = J_B/J_A$). The critical behaviour is characterized by the same critical exponents of the uniform model. For $C(q) > 1$, however, the uniform fixed point is totally unstable (as illustrated in figure 2(b)), which indicates a change in the character of the transition. From the condition $C(q) = 1$, we obtain the critical value $q = q_d = 4 + 2\sqrt{2}$ (where the subscript d stands for deterministic). For $q > q_d$, that corresponds to $C(q) > 1$, the uniform fixed point is fully unstable. The geometric fluctuations are irrelevant for $q < q_d$, as in the case of the Ising model ($q = 2$), but become relevant for $q > q_d$. It should be remarked that, as shown by Derrida and Gardner [11], the same value $q_r = 4 + 2\sqrt{2}$ (where r stands for random) corresponds to the crossover between uniform and disordered fixed points for a disordered ferromagnetic Potts model on the simple diamond hierarchical lattice we are discussing (see equation (3)).

Now we consider a Potts model on a *general* DHL, with m branches in parallel, each one of them with b bonds in series (and hence a chemical length b), and with ferromagnetic interactions according to the two-letter period-b substitution $(A, B) \to (A^{n_1}B^{b-n_1}, A^{n_2}B^{b-n_2})$, with $0 \leqslant n_1 < b$, $0 < n_2 \leqslant b$, and where the order of the letters A and B does not matter. This family of hierarchical structures includes the lattices that represent the Migdal–Kadanoff renormalization group approximations for this model on a d-dimensional hypercubic Bravais lattice (d coincides with their fractal dimension). The substitution matrix associated with the period-b sequence is given by

$$\mathbf{M} = \begin{pmatrix} n_1 & n_2 \\ b - n_1 & b - n_2 \end{pmatrix} \tag{9}$$

with eigenvalues $\lambda_1 = b$ and $\lambda_2 = n_1 - n_2$. Hence, from equation (4):

$$\omega = \frac{\ln|n_1 - n_2|}{\ln b}. \tag{10}$$

Using techniques of graph theory, as in the work of Essam and Tsallis [12], it is not difficult to write the recursion relations

$$t'_A = \frac{N(t_A, t_B; n_1)}{D(t_A, t_B; n_1)} \qquad \text{and} \qquad t'_B = \frac{N(t_A, t_B; n_2)}{D(t_A, t_B; n_2)} \tag{11}$$

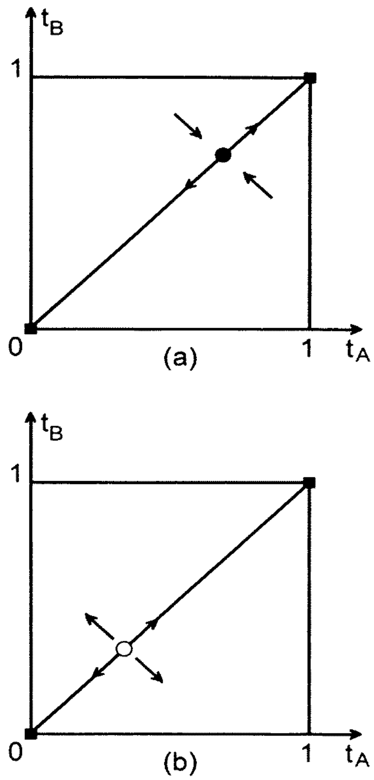

Figure 2. Schematic representations of the flow diagrams for the ferromagnetic Potts model in the (t_A, t_B) parameter space: (a) for $q < q_d$, and (b) for $q > q_d$. The arrows indicate the direction of the flow for consecutive (alternating) iterations when the smallest eigenvalue $\Lambda_2(q)$ of the map is positive (negative), i.e. when $n_1 > n_2$ $(n_1 < n_2)$. Squares, full circles, and open circles, represent fully stable, semistable and unstable fixed points, respectively. The diagonal $t_A = t_B$ is an invariant subspace under the renormalization group transformation.

where

$$N(t_A, t_B; n) = \sum_{l=1}^{m} \frac{F(q, G_{l+1})}{(q-1)} \; t_A^{nl} t_B^{(b-n)l} C_l^m \tag{12}$$

and

$$D(t_A, t_B; n) = 1 + \sum_{l=2}^{m} F(q, G_l) t_A^{nl} t_B^{(b-n)l} C_l^m \tag{13}$$

where G_l is the graph formed by l parallel edges, C_l^m is a combinatorial number and $F(q, G_l)$ is the flow polynomial [12] associated with G_l. For example, $F(q, G_2) = (q-1)$, $F(q, G_3) = (q-1)(q-2)$, $F(q, G_4) = (q-1)(q^2-3q+3)$, and we can use the deletion–contraction rule to write the recursion relation

$$F(q, G_l) = (q-1)^{l-1} - F(q, G_{l-1}). \tag{14}$$

From these equations, we can easily derive equations (6) and (7) for the simple diamond hierarchical lattice. For a general DHL, the fixed points in the two-parameter space include those of the uniform case (for which $t_A = t_B$). Again, besides the trivial fixed points, there is a non-trivial uniform fixed point, $0 < t_A = t_B = t_u^*(q) < 1$. As in the previous example, the linearization of the recursion relations in the neighbourhood of this uniform fixed point, $t_u^*(q)$, still leads to the same form of matrix relation given by equation (8), with $C(q) = \Lambda_1(q)/b$, where $\Lambda_1(q)$ is the thermal eigenvalue of the uniform model $(J_A = J_B > 0)$. In fact, the prefactor $C(q)$ can be calculated from the renormalized transmissivity $t'(t_1, t_2, \ldots, t_{mb})$ of the DHL under consideration,

$$C(q) = m \frac{\partial t'}{\partial t_i}\bigg|_{t_u^*(q)} \qquad (15)$$

where the ith bond ($i = 1, 2, \ldots, mb$) has a transmissivity t_i, and where $t_1 = t_2 = \cdots = t_{mb} = t_u^*(q)$. Due to the invariance of $t'(t_1, \ldots, t_{mb})$ under any permutation of the t_i's, all the mb derivatives $\partial t'/\partial t_i|_{t_u^*}$ are equal among themselves. Derrida *et al* [2] have shown that, if this symmetry condition holds for the quenched disordered Potts model on a hierarchical lattice, then we can use the Harris criterion, that is, disorder is relevant (irrelevant) when the critical exponent α_u of the uniform case is positive (negative). In the absence of this symmetry condition, the disorder is relevant for α_u above a negative critical value. In the symmetric case, disorder starts to become relevant at a critical number q_r of states, corresponding to the vanishing of α_u, such that

$$\frac{\partial t'}{\partial t_i}\bigg|_{t_u^*(q_r)} = \frac{1}{\sqrt{bm}}. \qquad (16)$$

For the aperiodic Potts model of this paper, the eigenvalues of the linearization of the recursion relations in the neighbourhood of $t_u^*(q)$ are $\Lambda_1(q) = \lambda_1 C(q) = bC(q)$ and $\Lambda_2(q) = \lambda_2 C(q) = (n_1 - n_2)C(q)$. Therefore, as $\Lambda_1 > 1$, the uniform fixed point becomes fully unstable for

$$|\Lambda_2(q)| = |n_1 - n_2| C(q) > 1. \qquad (17)$$

From equation (15), the number of states q_d associated with the onset of relevance of the geometrical fluctuations is given by

$$\frac{\partial t'}{\partial t_i}\bigg|_{t_u^*(q_d)} = \frac{1}{m|n_1 - n_2|}. \qquad (18)$$

Comparing equations (16) and (18), we see that q_r coincides with q_d if $b = m|n_1 - n_2|^2$.

Now we investigate the implications of the condition under which the non-trivial uniform fixed point becomes fully unstable. Let us consider the recursion relation associated with the uniform model ($J_A = J_B > 0$). From the linearization about the non-trivial fixed point, we have $\Lambda_1 = bC(q) = b^{y_t}$, with the thermal exponent [8, 11, 13] $y_t = D/(2 - \alpha_u)$, where $D = \ln(bm)/\ln b$ is the fractal dimension of the DHL. Therefore, $C(q) = b^{D/(2-\alpha_u)-1}$. From equation (10) we also have $|n_1 - n_2| = b^{\omega}$. Inserting the expressions for $C(q)$ and $|n_1 - n_2|$ into equation (17), we show that the geometric fluctuations become relevant for

$$\omega > 1 - \frac{D}{2 - \alpha_u} \qquad (19)$$

and irrelevant for $\omega < 1 - D/(2 - \alpha_u)$. Condition (19) reduces to the inequality $\alpha_u > 0$ if $\omega = 1 - D/2$, which occurs for $b = m|n_1 - n_2|^2$.

As an example, let us consider again the q-state Potts model on the simple diamond lattice ($b = 2$, $m = 2$) with aperiodic interactions according to the period-doubling substitution $(A, B) \to (AB, AA)$ (that is, with $n_1 = 1$ and $n_2 = 0$). As $\omega = 0$ and $D = 2$, the geometric fluctuations become relevant for $\alpha_u > 0$, which is identical to the criterion of Derrida and Gardner [11] for the relevance of disorder in the ferromagnetic Potts model on the simple diamond lattice. Also, $\alpha_u > 0$ is associated with $q > q_d = q_r = 4 + 2\sqrt{2}$.

To give another example, consider the q-state Potts model on a DHL with $b = 3$ bonds per branch and $m = 3$ branches (fractal dimension $D = 2$), and with ferromagnetic aperiodic interactions according to the two-letter substitution $(A, B) \to (ABB, AAA)$ (that is, $n_1 = 1$ and $n_2 = 3$, and hence $b \neq m|n_1 - n_2|^2$). As $\omega = \ln 2/\ln 3$, the geometric fluctuations become relevant for $\alpha_u > -2(\ln 2)/\ln(\frac{3}{2})$, that corresponds to $q > q_d = 0.226\,414\ldots$. Therefore, the critical behaviour of the Ising version of this model ($q = 2$) is drastically affected by the geometric fluctuations. However, quenched disorder is still irrelevant up to much bigger values of q (in this example, the crossover to a disordered fixed point only occurs for $q > q_r = 7.722\,361\ldots$).

Let us now mention that an alternative wandering exponent $\bar{\omega}$ can be defined in this problem in the following manner. Let us denote by $\bar{N}_A^{(n)}$ and $\bar{N}_B^{(n)}$ the respective numbers of letters A and B at the nth level of construction of the considered *hierarchical* lattice. One can define a matrix $\bar{M}$, which relates $\bar{N}_A^{(n)}$ and $\bar{N}_B^{(n)}$ with $\bar{N}_A^{(n-1)}$ and $\bar{N}_B^{(n-1)}$, similar to the definition of the substitution matrix M (equation (2)). One can easily show that $\bar{M} = m\,M$, $\bar{\lambda}_1 = m\bar{\lambda}_1$ and $\bar{\lambda}_2 = m\;\bar{\lambda}_2$ where $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are the eigenvalues of $\bar{M}$. Consequently, the total number of letters $\bar{N}^{(n)} = \bar{N}_A^{(n)} + \bar{N}_B^{(n)}$, at a large level n of the hierarchical level, has a subdominant term $\Delta\bar{N}^{(n)}$ which behaves asymptotically as $\Delta\bar{N}^{(n)} \sim (\bar{N}^{(n)})^{\bar{\omega}}$ where $\bar{\omega} = \ln|\bar{\lambda}_2|/\ln\bar{\lambda}_1 = 1 + (\omega - 1)/D$. One can, thus, rewrite equation (19) in terms of $\bar{\omega}$ as

$$\bar{\omega} > \frac{1 - \alpha_u}{2 - \alpha_u} \tag{20}$$

which—interestingly enough—formally coincides with Luck's criterion [3] for statistical-mechanical models (with aperiodicity in the coupling constants) on *Bravais* lattices and quasicrystals.

In conclusion, deterministic geometric fluctuations and random disorder are both capable of introducing drastic changes in the critical behaviour of a statistical model. We have established a criterion to check the relevance of geometric fluctuations in the critical behaviour of ferromagnetic Potts models. This criterion is *exact* for DHL and possibly a good approximation for (hypercubic) Bravais lattices. Geometrical and random fluctuations, however, are distinct phenomena. For example, in the case of the q-state Potts ferromagnet, the threshold for the onset of changes in the critical behaviour may occur at different values, $q_d \neq q_r$, in the deterministic and the random cases.

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References

[1] Harris A B 1974 *J. Phys. C: Solid State Phys.* **7** 1671
[2] Derrida B, Dickinson H and Yeomans J 1985 *J. Phys. A: Math. Gen.* **18** L53
[3] Luck J M 1993 *Europhys. Lett.* **24** 359
Luck J M 1993 *J. Stat. Phys.* **72** 417
[4] Shechtman D, Blech I, Gratias D and Cahn J W 1984 *Phys. Rev. Lett.* **53** 1951
Levine D and Steinhardt J 1984 *Phys. Rev. Lett.* **53** 2477
Merlin R, Bajema K, Clarke R, Juang F-Y and Bhattacharya P K 1985 *Phys. Rev. Lett.* **55** 1768
Bajema K and Merlin R 1987 *Phys. Rev.* B **36** 4555
[5] Grimm U and Baake M 1997 Aperiodic Ising models *The Mathematics of Long-range Aperiodic Order* ed R V Moody (Amsterdam: Kluwer) pp 199–237
Iglói F and Turban L 1996 *Phys. Rev. Lett.* **77** 1206
Berche B, Berche P E, Henkel M, Iglói F, Lajko P, Morgan S and Turban L 1995 *J. Phys. A: Math. Gen.* **28** L165
Iglói F and Turban L 1994 *Europhys. Lett.* **27** 91
Iglói F 1993 *J. Phys. A: Math. Gen.* **26** L703
Iglói F, Karevski D and Rieger H 1998 *Eur. Phys. J.* **B1** 513
Hermisson J, Grimm U and Baake M 1997 *J. Phys. A: Math. Gen.* **30** 7315
[6] Pinho S T R, Haddad T A S and Salinas S R 1997 *Braz. J. Phys.* **27** 567

Pinho S T R, Haddad T A S and Salinas S R 1997 *Preprint* cond-mat/9709303

[7] Berker A N and Ostlund S 1979 *J. Phys. C: Solid State Phys.* **12** 4961
Kaufman M and Griffiths R B 1981 *Phys. Rev.* B**24** 496
Griffiths R B and Kaufman M 1982 *Phys. Rev.* B **26** 5022
Melrose J R 1983 *J. Phys. A: Math. Gen.* **16** 1041
Melrose J R 1983 *J. Phys. A: Math. Gen.* **16** 3077

[8] Tsallis C 1985 *J. Phys. C: Solid State Phys.* **18** 6581

[9] Tsallis C and Levy S V F 1981 *Phys. Rev. Lett.* **47** 950

[10] Tsallis C and de Magalhães A C N 1996 *Phys. Rep.* **268** 305

[11] Derrida B and Gardner E 1984 *J. Phys. A: Math. Gen.* **17** 3223

[12] Essam J W and Tsallis C 1986 *J. Phys. A: Math. Gen.* **19** 409

[13] da Cruz H R and Stinchcombe R B 1986 *J. Phys. C: Solid State Phys.* **19** 3555